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Electromagnetic waves from any isolated source

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Abstract. A double-parameter approximation method is applied to the Einstein–Maxwell equations of general relativity for the study of electromagnetic waves from any isolated cohesive source. It is found that, in general, the source undergoes secular changes of mass and angular momentum at rates equal and opposite to those at which energy and angular momentum are carried away by the waves from the source, as calculated by the electromagnetic energy tensor.

1. Introduction

In a previous paper it was shown that a finite oscillating coherent linear distribution of charge emitting electromagnetic waves undergoes a permanent reduction of mass, accounting for the flow of radiation energy from the source as calculated by means of the electromagnetic energy tensor (Rotenberg 1966). One aim of the present work is to extend this result to any isolated cohesive source of electromagnetic waves. The main object, however, is to establish that a generally permanent change in angular momentum of the source occurs which is equal and opposite to the angular momentum transmitted from the source by electromagnetic radiation. In obtaining expressions representing the rates of variation in mass and angular momentum of the source as outgoing radiation, we shall use a double-parameter approximation method described in § 3, formally similar to that invented by Bonnor (1959) and appearing in the above-mentioned paper of Rotenberg (1966). As in the latter work, the approximation method will be applied to the metric tensor and then to the Einstein–Maxwell equations†

$$R_{ik} = -8\pi E_{ik} \qquad E^i_k = -F^{ia}F_{ka} + \tfrac{1}{4}\delta^i_k F^{ab}F_{ab}$$
$$F^{ia}{}_{;a} = 0 \qquad F_{ik} = \phi_{i,k} - \phi_{k,i} \tag{1.1}$$

for free space (Eddington 1924, §§ 73 and 77), where ϕ_i, F_{ik} and E_{ik} are the electromagnetic 4-potential, 4×4-field and energy tensors, respectively. To reduce calculations considerably, coordinates of the Sachs metric (introduced in § 4) will be employed for carrying out this method.

The exterior retarded multipole wave solution for ϕ_i of the linearized Einstein–Maxwell equations is given in § 2 for outgoing electromagnetic waves from any isolated coherent source; the solution is expressed in (pseudo-) Galilean coordinates referred to here as $x_i = (x, y, z, t) = (x_\alpha, t)$ with their origin as O. In § 3, the double-parameter

† In this paper, Latin indices range from 1 to 4, Greek indices from 1 to 3; the summation convention applies to both forms of indices. Comma subscripts indicate partial differentiation and semicolon subscripts denote covariant differentiation, with respect to the coordinate system used.

approximation method is outlined, and the metric invented by Sachs (1962) is presented in § 4, where also the exterior multipole wave solution for ϕ_i in the coordinates of this metric is derived from the one of § 2 in Galilean coordinates. The multipole wave solution in the Sachs metric is required in § 5 to calculate the components of the electromagnetic energy tensor in this metric and formulae for the rates at which energy and angular momentum of electromagnetic radiation are transmitted from the source. Finally, in § 6 the main results are obtained from the approximate Einstein–Maxwell equations: that there occur, in general, secular changes in mass and angular momentum of the source at rates equal and opposite to those at which energy and angular momentum are carried away from the source by radiation. The more lengthy calculations and some useful identities are relegated to the appendixes.

2. The retarded solution of the linearized Einstein–Maxwell equations†

For ϕ_i in Galilean coordinates $x_i = (x, y, z, t) = (x_\alpha, t)$, we present here, after introducing relevant notation, the exterior retarded multipole wave solution of the linearized form of the second pair of equations (1.1) or of

$$F^{ia}{}_{;a} = 4\pi J^i \qquad F_{ik} = \phi_{i,k} - \phi_{k,i} \tag{2.1}$$

in which J_i is the 4-current density of the source of the field. See appendix 1 for a derivation of this solution.

Let m be the total mass and e the total charge of the source, so that

$$m = \int_V T_{44}\,\mathrm{d}v \qquad e = \int_V J_4\,\mathrm{d}v \tag{2.2}$$

V being any space volume sufficient to include the source; and let a be a constant, with the dimensions of length, characterizing the extent of the source by representing, for example, the radius of gyration of the source averaged over all time. Let

$$I_{i:\sigma\rho\tau\ldots}(t) \stackrel{\mathrm{def}}{=} \int_V x_\sigma x_\rho x_\tau \ldots J_i(x_\alpha, t)\,\mathrm{d}v \qquad (\mathrm{d}v = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3) \tag{2.3}$$

be the moments at time t of the 4-current density J_i for the source about the coordinate planes $x_\alpha = 0$, so that these moments satisfy the conservation law

$$\eta^{ab} J_{a,b} = 0 \qquad \eta^{ik} = \eta_{ik} \stackrel{\mathrm{def}}{=} \mathrm{diag}(-1, -1, -1, +1) \tag{2.4}$$

for J_i. Introduce the *specific* moments, unaffected by any change in units of e or a, as

$$\begin{aligned} h_{\alpha:\sigma_1\sigma_2\ldots\sigma_s} &\stackrel{\mathrm{def}}{=} e^{-1} a^{-s-1} I_{\alpha:\sigma_1\sigma_2\ldots\sigma_s} \\ h_{4:\sigma_1\sigma_2\ldots\sigma_s} &\stackrel{\mathrm{def}}{=} e^{-1} a^{-s} I_{4:\sigma_1\sigma_2\ldots\sigma_s} \end{aligned} \tag{2.5}$$

† It will be assumed that, in the linear approximation to the Einstein–Maxwell equations, distance, time and mass retain their Newtonian significance.

and write

$$k_\alpha \overset{\text{def}}{=} h_{4:\alpha} \qquad k_{\alpha\beta} \overset{\text{def}}{=} h_{4:\alpha\beta} \qquad l_{\alpha\beta} \overset{\text{def}}{=} h_{\alpha:\beta}. \tag{2.6}$$

Finally, let $r \overset{\text{def}}{=} (x_\sigma x_\sigma)^{1/2}$ be the radial coordinate OP of the field point P having spherical polar coordinates (r, θ, ϕ), and write

$$n_\alpha \overset{\text{def}}{=} x_\alpha/r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{2.7}$$

Then, for outgoing waves, the exterior multipole wave solution of the linear approximation to equations (2.1) is (appendix 1)

$$\begin{aligned} \phi_\alpha &= e[-ar^{-1}k'_\alpha + a^2 n_\sigma(r^{-1}l'_{\alpha\sigma} + r^{-2}l_{\alpha\sigma}) + \mathrm{O}(a^3)] \\ \phi_4 &= e\{r^{-1} + an_\sigma(r^{-1}k'_\sigma + r^{-2}k_\sigma) + a^2[-r^{-1}n_\sigma n_\rho l'_{\sigma\rho} \\ &\quad + \tfrac{1}{2}(3n_\sigma n_\rho - \delta_{\sigma\rho})(-2r^{-2}l_{\sigma\rho} + r^{-3}k_{\sigma\rho})] + \mathrm{O}(a^3)\} \end{aligned} \tag{2.8}$$

where k_α, $k_{\alpha\beta}$, $l_{\alpha\beta}$ are to be evaluated at retarded time $u \overset{\text{def}}{=} t-r$ and a prime indicates differentiation with respect to the argument u.

In the solution (2.8) the 2^s *pole wave* ($s = 0, 1, 2, \ldots$) is the part involving ea^s; only the monopole contribution and the dipole and quadrupole wave contributions have been explicitly written in this multipole wave solution, since these will be sufficient to deduce the results of § 6.

3. The double-parameter approximation method

The source of the combined gravitational and electromagnetic field is characterized by the three parameters m, e and a introduced in § 2. So we shall assume that the metric tensor g_{ik} representing the external field can be expanded as a convergent triple-series in ascending powers of m, e and a:

$$g_{ik} = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{s=0}^{\infty} m^n e^p a^s \overset{(nps)}{g_{ik}} \tag{3.1}$$

where $\overset{(nps)}{g_{ik}}$ are independent of m, e or a. We now split up the right-hand side of equation (3.1) into suitable components, using the steps below. When this is achieved we shall find that we need confine attention only to a part involving a double power series in e and a, which is relevant to the present study of the electromagnetic waves from the source. We proceed to break up the expansion (3.1) in the following four stages:

$$\begin{aligned} g_{ik} &= \sum_{p=0}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(0ps)}{g_{ik}} + \sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\sum_{s=0}^{\infty} m^n e^p a^s \overset{(nps)}{g_{ik}} \\ &= \sum_{p=0}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(0ps)}{g_{ik}} + \sum_{n=1}^{\infty}\sum_{s=0}^{\infty} m^n a^s \overset{(n0s)}{g_{ik}} + \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^n e^p a^s \overset{(nps)}{g_{ik}} \\ &= \sum_{s=0}^{\infty} a^s \overset{(00s)}{g_{ik}} + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(0ps)}{g_{ik}} + \sum_{n=1}^{\infty}\sum_{s=0}^{\infty} m^n a^s \overset{(n0s)}{g_{ik}} + \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^n e^p a^s \overset{(nps)}{g_{ik}} \end{aligned}$$

so finally,

$$g_{ik} = \overset{(000)}{g_{ik}} + \sum_{s=1}^{\infty} a^s \overset{(00s)}{g_{ik}} + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(0ps)}{g_{ik}} + \sum_{n=1}^{\infty}\sum_{s=0}^{\infty} m^n a^s \overset{(n0s)}{g_{ik}}$$
$$+ \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^n e^p a^s \overset{(nps)}{g_{ik}}. \qquad (3.2)$$

The field depends essentially on the existence of the mass and charge of the source; if these are absent (ie, if $m = 0$, $e = 0$), the space–time will be flat and the constant a should, consequently, not appear on the right of equation (3.1)†. From this it follows that

$$\overset{(00s)}{g_{ik}} = 0 \qquad (s \geqslant 1) \qquad (3.3)$$

and the second, single summation, term on the right of equation (3.2) vanishes. The fourth, double summation, term on the right of equation (3.2) involves the parameters m and a and refers to the external gravitational field of the source, which is not our concern here. The last, triple summation, term on the right of equation (3.2) contains all the three parameters m, e and a and represents the interaction between the external gravitational and electromagnetic fields. Since both m and e are small in relativistic units, the leading effects of this interaction on the source are expected to be small compared with those of the purely gravitational or electromagnetic waves corresponding to the respective double summation terms on the right of equation (3.2).

From the foregoing considerations, the only contribution to g_{ik} in equation (3.2) that we need focus attention on is the purely electromagnetic contribution

$$g_{ik} = \overset{(00)}{g_{ik}} + \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{g_{ik}} \qquad (3.4)$$

in which $\overset{(00)}{g_{ik}}$ and $\overset{(ps)}{g_{ik}}$, independent of m, e or a, are condensed forms of $\overset{(000)}{g_{ik}}$ and $\overset{(0ps)}{g_{ik}}$, respectively; the isolated term $\overset{(00)}{g_{ik}}$ refers to flat space–time.

The reason for starting the summation with respect to p in equation (3.4) from $p = 2$, rather than $p = 1$ as in the corresponding double summation term in equation (3.2), is as follows. Use of the expansion (3.4) for g_{ik} together with equations (2.8) and the fourth of equations (1.1) in the second of equations (1.1) will result in a similar expansion for E_{ik}, namely

$$E_{ik} = \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{E_{ik}} \qquad (3.5)$$

in which $\overset{(ps)}{E_{ik}}$ do not involve m, e or a. In this expansion (3.5), the summation with respect to p begins with $p = 2$, and it is readily seen that this would still be the case even if the range of summation in the expansion (3.4) for g_{ik} were to be extended to include $p = 1$. So we may as well allow the summation with respect to p for g_{ik} to commence with $p = 2$ as in the expansion (3.5) for E_{ik}, since the metric depends in part on the electromagnetic field represented by E_{ik}.

† Pure gravitational and electromagnetic waves, those which possess no sources, are excluded from our discussion.

The double-series expansions (3.4) and (3.5), first introduced by Rotenberg (1966), will now be the basis of the double-parameter approximation method, used for studying the effects of electromagnetic waves on the mass and angular momentum of their source. Inserting these expansions in the first of equations (1.1) and equating the coefficients of $e^p a^s$ on both sides for each given pair p and s, we obtain ten second-order differential equations of the form

$$\Phi_{lm}(\overset{(ps)}{g}_{ik}) = \overset{(ps)}{\Psi}_{lm}(\overset{(qr)}{g}_{ik}) + \text{constant} \times \overset{(ps)}{E}_{lm} \qquad (2 \leqslant q \leqslant p-1, 0 \leqslant r \leqslant s) \quad (3.6)$$

to be referred to as the (ps) *approximation*. The left-hand sides, Φ_{lm}, are linear in $\overset{(ps)}{g}_{ik}$ and their derivatives; $\overset{(ps)}{\Psi}_{lm}$ on the right-hand sides are nonlinear in $\overset{(qr)}{g}_{ik}$ and their derivatives, determined from earlier approximations. Thus, besides the expressions in $\overset{(2s)}{E}_{ik}$, the $(2s)$ approximations involve only terms linear in $\overset{(2s)}{g}_{ik}$ and their derivatives; the nonlinear $\overset{(ps)}{\Psi}_{lm}$ do not appear in the $(2s)$ approximations. These approximations are the only ones considered in this paper. In fact, it is in the (22) approximation that there first appear terms generally representing secular changes in mass and angular momentum of the source at rates precisely equal and opposite to those at which energy and angular momentum of electromagnetic waves flow away from the source (§ 6). So our object, in § 6, is to find appropriate solutions of the $(2s)$ approximations ($s = 0, 1, 2$).

In conclusion, we denote the solution of the (ps) approximation as the (ps) *solution*; it is represented by the $\overset{(ps)}{g}_{ik}$ satisfying equations (3.6).

4. The Sachs metric

In order to solve the leading $(2s)$ approximations effectively, especially the (22) one, we shall use the metric of Sachs (1962), presented here in the form

$$\begin{aligned} ds^2 &= -r^2(B\, d\theta^2 - 2I \sin\theta\, d\theta\, d\phi + C \sin^2\theta\, d\phi^2) + D\, du^2 \\ &\quad + 2F\, dr\, du + 2Gr\, d\theta\, du + 2Jr \sin\theta\, d\phi\, du \qquad (4.1) \\ BC - I^2 &= 1 \end{aligned}$$

as in Rotenberg (1972a, b). In this, B, C, D, F, G, I and J are functions of the coordinates (r, θ, ϕ, u), of which the space-like coordinates (r, θ, ϕ) are the (pseudo-) spherical polar coordinates of the field point P, and the time-like coordinate $u \overset{\text{def}}{=} t - r$ is the (pseudo-) retarded time at P.

In coordinates of the Sachs metric, flat space–time is represented by

$$ds^2 = -r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + du^2 + 2\, dr\, du \qquad (4.2)$$

and the external Nordström solution takes the form

$$ds^2 = -r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + (1 - 2mr^{-1} + 4\pi e^2 r^{-2})du^2 + 2\, dr\, du \qquad (4.3)$$

as shown in Rotenberg (1971).

The coefficients of the metric (4.1) have expansions similar to the expansion (3.4), given by

$$
\begin{aligned}
-r^{-2}g_{22} &= B = 1+\sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{B}\\
-r^{-2}\operatorname{cosec}^2\theta g_{33} &= C = 1+\sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{C}\\
g_{44} &= D = 1+\sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{D}\\
g_{14} &= F = 1+\sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{F}\\
r^{-1}g_{24} &= G = \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{G}\\
r^{-2}\operatorname{cosec}\theta g_{23} &= I = \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{I}\\
r^{-1}\operatorname{cosec}\theta g_{34} &= J = \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{J}
\end{aligned}
\tag{4.4}
$$

where $\overset{(ps)}{B}, \overset{(ps)}{C}, \ldots, \overset{(ps)}{J}$ are functions of (r, θ, ϕ, u), and $\overset{(ps)}{B}, \overset{(ps)}{C}$ and $\overset{(ps)}{I}$ are connected by the second of equations (4.1). The isolated terms 1 on the extreme right-hand sides of equations (4.4) constitute the flat space–time metric (4.2), in which

$$
\overset{(00)}{g}_{22} = -r^2 \qquad \overset{(00)}{g}_{33} = -r^2\sin^2\theta \qquad \overset{(00)}{g}_{44} = 1 \qquad \overset{(00)}{g}_{14} = 1. \tag{4.5}
$$

The notation (4.4) will be adopted in § 6.

To obtain the external multipole wave solution (of the linear approximation to the second pair of equations (1.1)) for ϕ_ι in coordinates of the Sachs metric, we apply the coordinate transformation

$$
x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \qquad t = u+r \tag{4.6}
$$

to equations (2.8). With $n_{\sigma,2} \overset{\text{def}}{=} \partial n_\sigma/\partial\theta$, $n_{\sigma,3} \overset{\text{def}}{=} \partial n_\sigma/\partial\phi$, the result is

$$
\begin{aligned}
\phi_1 &= e\{r^{-1}+ar^{-2}n_\sigma k_\sigma + a^2[r^{-2}(-2n_\sigma n_\rho+\delta_{\sigma\rho})l_{\sigma\rho}+\tfrac{1}{2}r^{-3}(3n_\sigma n_\rho-\delta_{\sigma\rho})k_{\sigma\rho}]+\mathrm{O}(a^3)\}\\
\phi_2 &= e[-an_{\sigma,2}k'_\sigma + a^2 n_{\sigma,2}n_\rho(l'_{\sigma\rho}+r^{-1}l_{\sigma\rho})+\mathrm{O}(a^3)]\\
\phi_3 &= e[-an_{\sigma,3}k'_\sigma + a^2 n_{\sigma,3}n_\rho(l'_{\sigma\rho}+r^{-1}l_{\sigma\rho})+\mathrm{O}(a^3)]\\
\phi_4 &= e\{r^{-1}+an_\sigma(r^{-1}k'_\sigma+r^{-2}k_\sigma)+a^2[-r^{-1}n_\sigma n_\rho l'_{\sigma\rho}\\
&\quad +\tfrac{1}{2}(3n_\sigma n_\rho-\delta_{\sigma\rho})(-2r^{-2}l_{\sigma\rho}+r^{-3}k_{\sigma\rho})]+\mathrm{O}(a^3)\}
\end{aligned}
\tag{4.7}
$$

in which k_α, $k_{\alpha\beta}$, $l_{\alpha\beta}$ are functions of u and a prime denotes differentiation with respect to u†. This solution will be needed in the next section.

5. The fluxes of energy and angular momentum

To determine the rates of flux of energy and angular momentum we obtain the values for the components E_{ik} of the electromagnetic energy tensor in coordinates of the Sachs metric, to the required degree of approximation. By inserting equations (4.7) in the fourth of equations (1.1) and using the result and equations (3.4) and (4.5) in the second of equations (1.1) we find after considerable but straightforward calculation

$$E_{ik} = e^2(\overset{(20)}{E}_{ik} + a\,\overset{(21)}{E}_{ik} + a^2\,\overset{(22)}{E}_{ik} + \mathrm{O}(a^3)) + \mathrm{O}(e^3) \tag{5.1}$$

in which the nonzero $\overset{(20)}{E}_{ik}$ and $\overset{(21)}{E}_{ik}$ are given by

$$r^{-2}\overset{(20)}{E}_{22} = r^{-2}s^{-2}\overset{(20)}{E}_{33} = \overset{(20)}{E}_{44} = \overset{(20)}{E}_{14} = \tfrac{1}{2}r^{-4} \tag{5.2}$$

$$\begin{aligned}
r^{-2}\overset{(21)}{E}_{22} &= r^{-2}s^{-2}\overset{(21)}{E}_{33} = \overset{(21)}{E}_{44} = \overset{(21)}{E}_{14} = 2n_\sigma(r^{-4}k'_\sigma + r^{-5}k_\sigma)\\
r^{-1}\overset{(21)}{E}_{12} &= r^{-5}n_{\sigma,2}k_\sigma\\
r^{-1}s^{-1}\overset{(21)}{E}_{13} &= r^{-5}s^{-1}n_{\sigma,3}k_\sigma\\
r^{-1}\overset{(21)}{E}_{24} &= -n_{\sigma,2}(r^{-3}k''_\sigma + r^{-4}k'_\sigma)\\
r^{-1}s^{-1}\overset{(21)}{E}_{34} &= -s^{-1}n_{\sigma,3}(r^{-3}k''_\sigma + r^{-4}k'_\sigma)
\end{aligned} \tag{5.3}$$

(with $s \overset{\text{def}}{=} \sin\theta$), in which the nonzero $(10)\times(12)$ contributions to $\overset{(22)}{E}_{ik}$ are given by‡

$$\begin{aligned}
r^{-2}\overset{(22)}{E}_{22} &= r^{-2}s^{-2}\overset{(22)}{E}_{33} = \overset{(22)}{E}_{44} = \overset{(22)}{E}_{14}\\
&= \tfrac{1}{2}(3n_\sigma n_\rho - \delta_{\sigma\rho})(-2r^{-4}l'_{\sigma\rho} - 6r^{-5}l_{\sigma\rho} + 3r^{-6}k_{\sigma\rho})\\
r^{-1}\overset{(22)}{E}_{12} &= n_{\sigma,2}n_\rho[r^{-5}(-l_{\sigma\rho} - 2l_{\sigma\rho}) + 3r^{-6}k_{\sigma\rho}]\\
r^{-1}s^{-1}\overset{(22)}{E}_{13} &= s^{-1}n_{\sigma,3}n_\rho[r^{-5}(-l_{\sigma\rho} - 2l_{\rho\sigma}) + 3r^{-6}k_{\sigma\rho}]\\
r^{-1}\overset{(22)}{E}_{24} &= n_{\sigma,2}n_\rho[r^{-3}l''_{\sigma\rho} + r^{-4}(2l'_{\sigma\rho} + l'_{\rho\sigma}) + r^{-5}(2l_{\sigma\rho} + l_{\rho\sigma})]\\
r^{-1}s^{-1}\overset{(22)}{E}_{34} &= s^{-1}n_{\sigma,3}n_\rho[r^{-3}l''_{\sigma\rho} + r^{-4}(2l'_{\sigma\rho} + l'_{\rho\sigma}) + r^{-5}(2l_{\sigma\rho} + l_{\rho\sigma})]
\end{aligned} \tag{5.4}$$

† A sequence of additional transformations in the form

$$x^i = x^{*i} + e^p a^s \overset{(ps)}{\xi}{}^i(x^*) \qquad (x^i = r, \theta, \phi, u, p \geqslant 1, s \geqslant 0)$$

usually required to ensure that a metric in coordinates (r, θ, ϕ, u) satisfies the conditions $g_{11} = g_{12} = g_{13} = 0$, $g_{22}g_{33} - g_{23}g_{23} = r^4\sin^2\theta$ of the Sachs metric, would only introduce higher-magnitude terms in the solution (4.7), of order $e^{p+1}a^s$ $(p \geqslant 1, s \geqslant 0)$, the ϕ_i in equations (2.8) being of order e.

‡ The $(1q)\times(1r)$ contribution to E_{ik} here means the part of E_{ik} coming from the combination $\overset{(1q)}{F}_{ik} \times \overset{(1r)}{F}_{ik}$ in the second of equations (1.1), $\overset{(ps)}{F}_{ik}$ denoting the coefficient of $e^p a^s$ in F_{ik}.

and in which the (11) × (11) contributions to $\overset{(22)}{E}_{ik}$ are given by

$$\overset{(22)}{E}_{11} = r^{-6}(k_\sigma k_\sigma - n_\sigma n_\rho k_\sigma k_\rho)$$

$$\begin{aligned}
r^{-2}\overset{(22)}{E}_{22} = {} & r^{-4}[k_\sigma k''_\sigma + n_\sigma n_\rho(-k_\sigma k''_\rho + 2k'_\sigma k'_\rho) - 2n_{\sigma,2}n_{\rho,2}k_\sigma k''_\rho] \\
& + r^{-5}[\tfrac{1}{2}(k_\sigma k_\sigma)' + \tfrac{3}{2}n_\sigma n_\rho(k_\sigma k_\rho)' - n_{\sigma,2}n_{\rho,2}(k_\sigma k_\rho)'] \\
& + r^{-6}(\tfrac{1}{2}k_\sigma k_\sigma + \tfrac{3}{2}n_\sigma n_\rho k_\sigma k_\rho - n_{\sigma,2}n_{\rho,2}k_\sigma k_\rho)
\end{aligned}$$

$$\begin{aligned}
r^{-2}s^{-2}\overset{(22)}{E}_{33} = {} & r^{-4}[-k_\sigma k''_\sigma + n_\sigma n_\rho(k_\sigma k''_\rho + 2k'_\sigma k'_\rho) + 2n_{\sigma,2}n_{\rho,2}k_\sigma k''_\rho] \\
& + r^{-5}[-\tfrac{1}{2}(k_\sigma k_\sigma)' + \tfrac{5}{2}n_\sigma n_\rho(k_\sigma k_\rho)' + n_{\sigma,2}n_{\rho,2}(k_\sigma k_\rho)'] \\
& + r^{-6}(-\tfrac{1}{2}k_\sigma k_\sigma + \tfrac{5}{2}n_\sigma n_\rho k_\sigma k_\rho + n_{\sigma,2}n_{\rho,2}k_\sigma k_\rho)
\end{aligned}$$

$$\begin{aligned}
\overset{(22)}{E}_{44} = {} & r^{-2}(k''_\sigma k''_\sigma - n_\sigma n_\rho k''_\sigma k''_\rho) + r^{-3}[(k'_\sigma k'_\sigma)' - n_\sigma n_\rho(k'_\sigma k'_\rho)'] \\
& + r^{-4}[\tfrac{1}{2}(k_\sigma k_\sigma)'' + n_\sigma n_\rho(-k_\sigma k''_\rho + k'_\sigma k'_\rho)] + r^{-5}[\tfrac{1}{2}(k_\sigma k_\sigma)' + \tfrac{3}{2}n_\sigma n_\rho(k_\sigma k_\rho)'] \qquad (5.5) \\
& + r^{-6}(\tfrac{1}{2}k_\sigma k_\sigma + \tfrac{3}{2}n_\sigma n_\rho k_\sigma k_\rho)
\end{aligned}$$

$$r^{-1}\overset{(22)}{E}_{12} = 2n_{\sigma,2}n_\rho(r^{-5}k_\sigma k'_\rho + r^{-6}k_\sigma k_\rho)$$

$$\overset{(22)}{E}_{14} = 2r^{-4}n_\sigma n_\rho k'_\sigma k'_\rho + 2r^{-5}n_\sigma n_\rho(k_\sigma k_\rho)' + r^{-6}(\tfrac{1}{2}k_\sigma k_\sigma + \tfrac{3}{2}n_\sigma n_\rho k_\sigma k_\rho)$$

$$r^{-1}\overset{(22)}{E}_{24} = -2n_\sigma n_{\rho,2}[r^{-3}k'_\sigma k''_\rho + r^{-4}(k_\sigma k'_\rho)' + r^{-5}k_\sigma k'_\rho]$$

$$r^{-1}s^{-1}\overset{(22)}{E}_{13} = 2s^{-1}n_{\sigma,3}n_\rho(r^{-5}k_\sigma k'_\rho + r^{-6}k_\sigma k_\rho)$$

$$r^{-2}s^{-1}\overset{(22)}{E}_{23} = -s^{-1}(n_{\sigma,2}n_{\rho,3} + n_{\sigma,3}n_{\rho,2})[r^{-4}k_\sigma k''_\rho + \tfrac{1}{2}r^{-5}(k_\sigma k_\rho)' + \tfrac{1}{2}r^{-6}k_\sigma k_\rho]$$

$$r^{-1}s^{-1}\overset{(22)}{E}_{34} = -2s^{-1}n_\sigma n_{\rho,3}[r^{-3}k'_\sigma k''_\rho + r^{-4}(k_\sigma k'_\rho)' + r^{-5}k_\sigma k'_\rho].$$

During the derivation of the above formulae and in § 6, it is helpful at times to consult the identities of appendix 5 to avoid excessive calculation.

As shown in appendix 2, the rates of total outward flow of energy and angular momentum from the source are given, in the linear approximation, by

$$\frac{\mathrm{d}}{\mathrm{d}t}(E) = \lim_{r\to\infty} r^2 \int (E^{11} + E^{14})\,\mathrm{d}\Omega \qquad (5.6)$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(M^{\mathrm{G}}_1) &= \lim_{r\to\infty} r^4 \int (-E^{12}\sin\phi - E^{13}sc\cos\phi)\,\mathrm{d}\Omega \\
\frac{\mathrm{d}}{\mathrm{d}t}(M^{\mathrm{G}}_2) &= \lim_{r\to\infty} r^4 \int (E^{12}\cos\phi - E^{13}sc\sin\phi)\,\mathrm{d}\Omega \qquad (5.7) \\
\frac{\mathrm{d}}{\mathrm{d}t}(M^{\mathrm{G}}_3) &= \lim_{r\to\infty} r^4 \int E^{13}s^2\,\mathrm{d}\Omega.
\end{aligned}$$

Here, the superscript G in M^{G}_α indicates that the three components M^{G}_α of the angular momentum are measured with respect to Galilean coordinates, and

$$\int f \, d\Omega \stackrel{\text{def}}{=} \int_0^{2\pi} \int_0^{\pi} f \sin\theta \, d\theta \, d\phi. \tag{5.8}$$

Making use of the above formulae and equations (3.4), (4.5), (5.1) to (5.5) and (2.7) leads (in a fairly straightforward manner—see appendix 2) to the following expressions for the rates of outward flow of energy and angular momentum from the source:

$$\frac{d}{dt}(E) = e^2(\tfrac{8}{3}\pi a^2 k''_\sigma k''_\sigma + O(a^3)) + O(e^3) \tag{5.9}$$

$$\frac{d}{dt}(M^G_\alpha) = e^2(\tfrac{8}{3}\pi a^2 \epsilon_{\alpha\sigma\rho} k'_\sigma k''_\rho + O(a^3)) + O(e^3) \tag{5.10}$$

in which $\epsilon_{\alpha\beta\gamma}$ is the permutation symbol. These are of order e^2a^2; consequently we expect terms to appear in the (22) approximation which indicate that the source itself undergoes variations in mass and angular momentum at rates equal and opposite to the above rates (5.9) and (5.10) specified by the leading, e^2a^2, parts. This is confirmed in the next section.

6. The second approximations. Secular changes in mass and angular momentum of the source in the (22) approximation

Each (2*s*) approximation is given in form by equations (A.26) to (A.35) of appendix 3 with the quantities $\overset{(2s)}{P}, \overset{(2s)}{Q}, \ldots, \overset{(2s)}{W}$ on the right specified as

$$\begin{aligned}
&\overset{(2s)}{P} = \alpha \overset{(2s)}{E}_{11} \quad &&\overset{(2s)}{Q} = \alpha r^{-2} \overset{(2s)}{E}_{22} \quad &&\overset{(2s)}{R} = \alpha r^{-2} s^{-2} \overset{(2s)}{E}_{33} \quad &&\overset{(2s)}{S} = \alpha \overset{(2s)}{E}_{44} \\
&\overset{(2s)}{L} = \alpha r^{-1} \overset{(2s)}{E}_{12} \quad &&\overset{(2s)}{M} = \alpha \overset{(2s)}{E}_{14} \quad &&\overset{(2s)}{N} = \alpha r^{-1} \overset{(2s)}{E}_{24} \quad &&\overset{(2s)}{U} = \alpha r^{-1} s^{-1} \overset{(2s)}{E}_{13} \\
&\overset{(2s)}{V} = \alpha r^{-2} s^{-1} \overset{(2s)}{E}_{23} \quad &&\overset{(2s)}{W} = \alpha r^{-1} s^{-1} \overset{(2s)}{E}_{34}
\end{aligned} \tag{6.1}$$

with

$$\alpha \stackrel{\text{def}}{=} -16\pi. \tag{6.2}$$

The solution of each (2*s*) approximation can be determined by means of equations (A.36) to (A.42) through the technique outlined in appendix 3 immediately after these equations.

On insertion of the formulae (5.2) to (5.5) one by one in equations (6.1) and on use of equations (A.36) to (A.42) as explained in appendix 3, the following approximate solutions can eventually be found.

The (20) *(Nordström) solution*

$$\overset{(20)}{D} = 4\pi r^{-2}. \tag{6.3}$$

The (21) *solution*

$$\begin{aligned}
&\overset{(21)}{D} = \alpha n_\sigma(-\tfrac{2}{3} r^{-2} k'_\sigma - \tfrac{1}{2} r^{-3} k_\sigma) \qquad \overset{(21)}{G} = \alpha n_{\sigma,2}(\tfrac{1}{3} r^{-2} k'_\sigma - \tfrac{1}{4} r^{-3} k_\sigma) \\
&\overset{(21)}{J} = \alpha s^{-1} n_{\sigma,3}(\tfrac{1}{3} r^{-2} k'_\sigma - \tfrac{1}{4} r^{-3} k_\sigma).
\end{aligned} \tag{6.4}$$

The (22) *solution corresponding to the* (10) × (12) *contribution to* E_{ik}

$$\begin{aligned}
\overset{(22)}{B} &= -\overset{(22)}{C} = \alpha S_{\sigma\rho}(-\tfrac{1}{12}r^{-3}l_{\sigma\rho}-\tfrac{1}{8}r^{-4}k_{\sigma\rho}) \\
\overset{(22)}{D} &= \alpha(3n_\sigma n_\rho-\delta_{\sigma\rho})(\tfrac{1}{3}r^{-2}l'_{\sigma\rho}+\tfrac{2}{3}r^{-3}l_{\sigma\rho}-\tfrac{1}{4}r^{-4}k_{\sigma\rho}) \\
\overset{(22)}{G} &= \alpha n_{\sigma,2}n_\rho[-\tfrac{1}{3}r^{-2}l'_{\sigma\rho}+r^{-3}(\tfrac{1}{8}l_{\sigma\rho}+\tfrac{3}{8}l_{\rho\sigma})-\tfrac{1}{2}r^{-4}k_{\sigma\rho}] \\
\overset{(22)}{I} &= \alpha s^{-1}Z_{\sigma\rho}(-\tfrac{1}{12}r^{-3}l_{\sigma\rho}-\tfrac{1}{8}r^{-4}k_{\sigma\rho}) \\
\overset{(22)}{J} &= \alpha s^{-1}n_{\sigma,3}n_\rho[-\tfrac{1}{3}r^{-2}l'_{\sigma\rho}+r^{-3}(\tfrac{1}{8}l_{\sigma\rho}+\tfrac{3}{8}l_{\rho\sigma})-\tfrac{1}{2}r^{-4}k_{\sigma\rho}].
\end{aligned} \tag{6.5}$$

The (22) *solution corresponding to the* (11) × (11) *contribution to* E_{ik}

$$\begin{aligned}
\overset{(22)}{B} &= -\overset{(22)}{C} = \alpha S_{\sigma\rho}[-\tfrac{1}{2}r^{-1}\tilde{E}_{\sigma\rho}+\tfrac{1}{8}r^{-3}(k_\sigma k_\rho)'] \\
\overset{(22)}{D} &= \alpha\{r^{-1}[-\tfrac{1}{3}\tilde{I}_{\sigma\sigma}+(n_\sigma n_\rho-\tfrac{1}{3}\delta_{\sigma\rho})(k'_\sigma k'_\rho)']+r^{-2}(\tfrac{1}{2}n_\sigma n_\rho-\tfrac{1}{2}\delta_{\sigma\rho})k'_\sigma k'_\rho \\
&\quad +r^{-3}(-\tfrac{1}{8}n_\sigma n_\rho-\tfrac{1}{8}\delta_{\sigma\rho})(k_\sigma k_\rho)'-\tfrac{1}{4}r^{-4}n_\sigma n_\rho k_\sigma k_\rho\} \\
\overset{(22)}{F} &= \alpha[\tfrac{1}{16}r^{-4}(-n_\sigma n_\rho+\delta_{\sigma\rho})k_\sigma k_\rho] \\
\overset{(22)}{G} &= \alpha n_\sigma n_{\rho,2}[r^{-1}\tilde{E}_{\sigma\rho}+r^{-2}(\tfrac{1}{3}\tilde{H}_{\sigma\rho}+k'_\sigma k'_\rho)+r^{-3}(\tfrac{3}{8}k_\sigma k'_\rho-\tfrac{1}{8}k'_\sigma k_\rho)-\tfrac{1}{8}r^{-4}k_\sigma k_\rho] \\
\overset{(22)}{I} &= \alpha s^{-1}Z_{\sigma\rho}[-\tfrac{1}{2}r^{-1}\tilde{E}_{\sigma\rho}+\tfrac{1}{8}r^{-3}(k_\sigma k_\rho)'] \\
\overset{(22)}{J} &= \alpha s^{-1}n_\sigma n_{\rho,3}[r^{-1}\tilde{E}_{\sigma\rho}+r^{-2}(\tfrac{1}{3}\tilde{H}_{\sigma\rho}+k'_\sigma k'_\rho)+r^{-3}(\tfrac{3}{8}k_\sigma k'_\rho-\tfrac{1}{8}k'_\sigma k_\rho)-\tfrac{1}{8}r^{-4}k_\sigma k_\rho].
\end{aligned} \tag{6.6}$$

In the above equations (6.5) and (6.6) the following notations apply:

$$\begin{aligned}
S_{\alpha\beta} &\overset{\text{def}}{=} \delta_{\alpha\beta}-n_\alpha n_\beta-2n_{\alpha,2}n_{\beta,2} = -n_{\alpha,2}n_{\beta,2}+s^{-2}n_{\alpha,3}n_{\beta,3} \\
Z_{\alpha\beta} &\overset{\text{def}}{=} n_{\alpha,2}n_{\beta,3}+n_{\alpha,3}n_{\beta,2} = (n_\alpha n_\beta)_{,23}-(n_\alpha n_\beta)_{,3}\cot\theta
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
\tilde{I}_{\alpha\beta} &\overset{\text{def}}{=} \int_{-\infty}^{u} k''_\alpha k''_\beta\,\mathrm{d}u \qquad \tilde{E}_{\alpha\beta} \overset{\text{def}}{=} \tfrac{1}{6}\tilde{I}_{\alpha\beta}-\tfrac{1}{3}(k'_\alpha k'_\beta)' \\
\tilde{H}_{\alpha\beta} &\overset{\text{def}}{=} \int_{-\infty}^{u} (k'_\alpha k''_\beta-k''_\alpha k'_\beta)\,\mathrm{d}u.
\end{aligned} \tag{6.8}$$

During the derivation of the solutions (6.4) to (6.6), the identities of appendix 5, involving n_α, have been helpful. Those could likewise be usefully employed in the verification of these solutions in the approximate field equations (A.26) to (A.35).

The solutions (6.3) to (6.5) were obtained without the use of the five functions (A.43) of integration: these functions were put equal to zero. However, towards obtaining the solution (6.6) the value

$$\chi = \alpha[(-\tfrac{1}{2}n_\sigma n_\rho-\tfrac{1}{6}\delta_{\sigma\rho})\tilde{I}_{\sigma\rho}+(2n_\sigma n_\rho-\tfrac{2}{3}\delta_{\sigma\rho})(k'_\sigma k'_\rho)'] \tag{6.9}$$

had to be assigned to the second function χ of integration, while the other four were ignored. Furthermore, the complementary solution

$$\overset{(22)}{G} = \alpha n_\sigma n_{\rho,2}(\tfrac{1}{2}r^{-1}\tilde{H}'_{\sigma\rho}+\tfrac{1}{3}r^{-2}\tilde{H}_{\sigma\rho}) \tag{6.10}$$

of the differential equation (A.38), satisfying $\square''\overset{(22)}{G} = 0$, had to be used. The omission of these crucial steps would result in a solution which, unlike equations (6.6), does not satisfy the regularity conditions for all θ, ϕ, u and all $r > 0$.

The linearized gravitational field outside a rigid spherical body of mass m rotating with angular momentum M^{G}_α is represented by the following Sachs metric, accurate to the quadrupole and r^{-2} terms and to be referred to as the *Schwarzschild angular momentum* metric:

$$\begin{aligned} \mathrm{d}s^2 = & -r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)+(1-2mr^{-1})\,\mathrm{d}u^2+2\,\mathrm{d}r\,\mathrm{d}u \\ & +2r^{-2}n_\sigma n_{\rho,2}A_{\sigma\rho}(2r\,\mathrm{d}\theta\,\mathrm{d}u)+2r^{-2}s^{-1}n_\sigma n_{\rho,3}A_{\sigma\rho}(2r\sin\theta\,\mathrm{d}\phi\,\mathrm{d}u) \end{aligned} \tag{6.11}$$

(see appendix 4); here $A_{\alpha\beta}$ is the skew angular momentum tensor (in Galilean coordinates) in which

$$M^{\mathrm{G}}_\alpha = (A_{23}, A_{31}, A_{12}). \tag{6.12}$$

Thus, in the determination of the secular changes in mass and angular momentum of the source, it is sufficient to seek secular changes in r^{-1} and r^{-2}, respectively, in the (2s) solutions over the period of motion of the source. Accordingly, disregarding all periodic terms and terms of order r^{-3} or higher in the solutions (6.3) to (6.6), we are left with the solution (6.6) containing the following integral terms (on account of the second of equations (6.8)), which generally represent secular changes in the metric:

$$\begin{aligned} &\overset{(22)}{B} = -\overset{(22)}{C} = -\tfrac{1}{12}\alpha r^{-1}S_{\sigma\rho}\tilde{I}_{\sigma\rho} \qquad \overset{(22)}{D} = -\tfrac{1}{3}\alpha r^{-1}\tilde{I}_{\sigma\sigma} \\ &\overset{(22)}{G} = \alpha n_\sigma n_{\rho,2}(\tfrac{1}{6}r^{-1}\tilde{I}_{\sigma\rho}+\tfrac{1}{3}r^{-2}\tilde{H}_{\sigma\rho}) \\ &\overset{(22)}{I} = -\tfrac{1}{12}\alpha r^{-1}s^{-1}Z_{\sigma\rho}\tilde{I}_{\sigma\rho} \qquad \overset{(22)}{J} = \alpha s^{-1}n_\sigma n_{\rho,3}(\tfrac{1}{6}r^{-1}\tilde{I}_{\sigma\rho}+\tfrac{1}{3}r^{-2}\tilde{H}_{\sigma\rho}). \end{aligned} \tag{6.13}$$

To obtain secular changes in mass and angular momentum of the source from this solution (6.13), we would require, as a temporary measure, that the source be in motion for only a given finite interval $0 \leqslant u \leqslant T$, say. The purpose of this is to allow the metric fields before $u = 0$ and after $u = T$ to be stationary and a comparison to be made between the two metrics. It is quite conceivable for the source to be provided with a component acting as a mechanism to start and stop the motion of the source rapidly but smoothly just before $u = 0$ and after $u = T$, respectively, as long as the motion does not involve any total angular momentum. However, it is desirable for the motion to be as general as possible; and to impart angular momentum to the source, which is to be in effect during the specified period $0 \leqslant u \leqslant T$ only, although generally impracticable, is theoretically possible. For example, the mechanism for producing such motion in a rigid charged body could be an insulated ringed device similar to that of the rotating rod discussed in Rotenberg (1972b). Only the ring would be in motion before $u = 0$ and after $u = T$ and the initial and final metrics would still be stationary. We shall assume that any prescribed motion in a general isolated cohesive source can be created somehow by an appropriate mechanism, without affecting the stationary character of the initial and final fields. Towards the end of this section it will be found that this is only a temporary arrangement, a thought experiment specially devised to compute mathematically the rates of changes in mass and angular momentum of the source; and concern about how to start and end the motion of the source in the most general way is not really necessary.

Now, for $u < 0$ (before the motion of the source), the solution (6.13) vanishes; so, as expected, there is nothing of order r^{-1} and r^{-2} to add to the initial stationary field. For $u > T$ (after the motion of the source), the solution becomes

$$\begin{aligned}
&\overset{(22)}{B} = -\overset{(22)}{C} = -\tfrac{1}{12}\alpha r^{-1} S_{\sigma\rho} i_{\sigma\rho} \qquad \overset{(22)}{D} = -\tfrac{1}{3}\alpha r^{-1} i_{\sigma\sigma} \\
&\overset{(22)}{G} = \alpha n_\sigma n_{\rho,2}(\tfrac{1}{6} r^{-1} i_{\sigma\rho} + \tfrac{1}{3} r^{-2} h_{\sigma\rho}) \\
&\overset{(22)}{I} = -\tfrac{1}{12}\alpha r^{-1} s^{-1} Z_{\sigma\rho} i_{\sigma\rho} \qquad \overset{(22)}{J} = \alpha s^{-1} n_\sigma n_{\rho,3}(\tfrac{1}{6} r^{-1} i_{\sigma\rho} + \tfrac{1}{3} r^{-2} h_{\sigma\rho})
\end{aligned} \tag{6.14}$$

where

$$\begin{aligned}
i_{\alpha\beta} &\overset{\text{def}}{=} \tilde{I}_{\alpha\beta}(T) = \int_0^T k''_\alpha k''_\beta \, \mathrm{d}u \\
h_{\alpha\beta} &\overset{\text{def}}{=} \tilde{H}_{\alpha\beta}(T) = \int_0^T (k'_\alpha k''_\beta - k''_\alpha k'_\beta) \, \mathrm{d}u
\end{aligned} \tag{6.15}$$

by virtue of the first and third of equations (6.8). The coordinate transformation

$$\begin{aligned}
r &= r^* + e^2 a^2 \overset{(22)}{\alpha}(\theta^*, \phi^*) & \overset{(22)}{\alpha}(\theta, \phi) &= \tfrac{1}{24}\alpha(-3 n_\sigma n_\rho + \delta_{\sigma\rho}) i_{\sigma\rho} \\
\theta &= \theta^* + e^2 a^2 \overset{(22)}{\beta}(r^*, \theta^*, \phi^*) & \overset{(22)}{\beta}(r, \theta, \phi) &= -\tfrac{1}{24}\alpha r^{-1}(n_\sigma n_\rho)_{,2} i_{\sigma\rho} \\
\phi &= \phi^* + e^2 a^2 \overset{(22)}{\gamma}(r^*, \theta^*, \phi^*) & \overset{(22)}{\gamma}(r, \theta, \phi) &= -\tfrac{1}{24}\alpha r^{-1} s^{-2}(n_\sigma n_\rho)_{,3} i_{\sigma\rho} \\
u &= u^* + e^2 a^2 \overset{(22)}{\delta}(\theta^*, \phi^*) & \overset{(22)}{\delta}(\theta, \phi) &= \tfrac{1}{24}\alpha(n_\sigma n_\rho - \delta_{\sigma\rho}) i_{\sigma\rho}
\end{aligned} \tag{6.16}$$

reduces the solution (6.14) to the form

$$\overset{(22)}{D} = -\tfrac{1}{3}\alpha r^{-1} i_{\sigma\sigma} \qquad \overset{(22)}{G} = \tfrac{1}{3}\alpha r^{-2} n_\sigma n_{\rho,2} h_{\sigma\rho} \qquad \overset{(22)}{J} = \tfrac{1}{3}\alpha r^{-2} s^{-1} n_\sigma n_{\rho,3} h_{\sigma\rho} \tag{6.17}$$

with the asterisks omitted, while the lower (ps) solutions are unaffected and the conditions of the Sachs metric remain satisfied. This solution (6.17) can be combined with the Schwarzschild angular momentum metric (6.11) to form the following metric of the same type representing the final stationary field:

$$\begin{aligned}
\mathrm{d}s^2 = &-r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2) + [1 - 2(m + \Delta m) r^{-1}] \, \mathrm{d}u^2 + 2 \, \mathrm{d}r \, \mathrm{d}u \\
&+ 2 r^{-2} n_\sigma n_{\rho,2}(A_{\sigma\rho} + \Delta A_{\sigma\rho})(2r \, \mathrm{d}\theta \, \mathrm{d}u) \\
&+ 2 r^{-2} s^{-1} n_\sigma n_{\rho,3}(A_{\sigma\rho} + \Delta A_{\sigma\rho})(2r \sin\theta \, \mathrm{d}\phi \, \mathrm{d}u)
\end{aligned} \tag{6.18}$$

with

$$\begin{aligned}
\Delta m &= \tfrac{1}{6}\alpha e^2 a^2 i_{\sigma\sigma} = -\tfrac{8}{3}\pi e^2 a^2 \int_0^T k''_\sigma k''_\sigma \, \mathrm{d}u \\
\Delta A_{\sigma\rho} &= \tfrac{1}{6}\alpha e^2 a^2 h_{\sigma\rho} = -\tfrac{8}{3}\pi e^2 a^2 \int_0^T (k'_\sigma k''_\rho - k''_\sigma k'_\rho) \, \mathrm{d}u
\end{aligned} \tag{6.19}$$

by virtue of the notations (6.2) and (6.15). These quantities clearly show up as corrections to the mass m and the angular momentum $A_{\sigma\rho}$ of the source. So, during the period

$0 \leqslant u \leqslant T$ of its motion the source undergoes changes in mass and angular momentum in the (22) approximation by the amounts

$$\begin{aligned} m(T)-m(0) &= -\tfrac{8}{3}\pi e^2 a^2 \int_0^T k''_\sigma k''_\sigma \,\mathrm{d}u+\mu \\ M^{\mathrm{G}}_\alpha(T)-M^{\mathrm{G}}_\alpha(0) &= -\tfrac{8}{3}\pi e^2 a^2 \epsilon_{\alpha\sigma\rho} \int_0^T k'_\sigma k''_\rho \,\mathrm{d}u+\mu_\alpha \end{aligned} \tag{6.20}$$

in the notation (6.12). Here we have included the contributions μ and μ_α to these changes due to the rapid but smooth transitions for the source from rest to motion and vice versa, only to show in the next paragraph that μ and μ_α can be made arbitrarily small by reducing the transitory periods sufficiently.

It would have been more accurate to add the short transitory periods to the period of the prescribed motion of the source and to take the augmented period as the range for the definite integrals on the right of equations (6.20). To compensate for this we have included the correction terms μ and μ_α on the right of equations (6.20), which are the contributions of these definite integrals over the period τ representing the transitory intervals combined, and which may be written as

$$\mu = -\tfrac{8}{3}\pi e^2 a^2 \int_\tau k''_\sigma k''_\sigma \,\mathrm{d}u \qquad \mu_\alpha = -\tfrac{8}{3}\pi e^2 a^2 \epsilon_{\alpha\sigma\rho} \int_\tau k'_\sigma k''_\rho \,\mathrm{d}u. \tag{6.21}$$

By improving the effectiveness of the stop-and-go mechanism sufficiently, we can reduce the combined transitory period sufficiently so as to render the quantities μ and μ_α in equations (6.21) so small as to justify these as before.

Suppose we allowed the motion of the source to proceed a little longer, over a period ΔT. Then we could calculate the derivatives $\mathrm{d}m(T)/\mathrm{d}T$ and $\mathrm{d}M^{\mathrm{G}}_\alpha(T)/\mathrm{d}T$ by subtracting the above effects (6.20) over the original period T from the corresponding effects over the extended period $T+\Delta T$, divide by ΔT and let ΔT tend to zero. Disregarding μ and μ_α in the process, as justified by the foregoing argument, we would find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}T}m(T) &= -\frac{8}{3}\pi e^2 a^2 \frac{\mathrm{d}}{\mathrm{d}T}\int_0^T k''_\sigma k''_\sigma \,\mathrm{d}u = -\frac{8}{3}\pi e^2 a^2 k''_\sigma(T) k''_\sigma(T) \\ \frac{\mathrm{d}}{\mathrm{d}T}M^{\mathrm{G}}_\alpha(T) &= -\frac{8}{3}\pi e^2 a^2 \epsilon_{\alpha\sigma\rho}\frac{\mathrm{d}}{\mathrm{d}T}\int_0^T k'_\sigma k''_\rho \,\mathrm{d}u = -\frac{8}{3}\pi e^2 a^2 \epsilon_{\alpha\sigma\rho} k'_\sigma(T) k''_\rho(T). \end{aligned} \tag{6.22}$$

This is true for any assigned value for T; so replacing T by u we have

$$\frac{\mathrm{d}}{\mathrm{d}u}m(u) = -\frac{8}{3}\pi e^2 a^2 k''_\sigma k''_\sigma \qquad \frac{\mathrm{d}}{\mathrm{d}u}M^{\mathrm{G}}_\alpha(u) = -\frac{8}{3}\pi e^2 a^2 \epsilon_{\alpha\sigma\rho} k'_\sigma k''_\rho \tag{6.23}$$

in which the argument of k_α is u. Hence we may conclude that, even if the motion of the source is allowed to carry on indefinitely without any attached device to start and stop its motion, the source will suffer secular changes in mass and angular momentum in the (22) approximation at the rates specified by the formulae (6.23). The requirement that the source be in motion for a finite period has now been relaxed, and equipping the source with a stop-and-go device may now be considered as a thought experiment.

As expected, the above effects (6.23) account for the (22) contributions to the rates (5.9) and (5.10) of radiation energy and angular momentum flow from the source.

Appendix 1. The linearized retarded solution for the electromagnetic potential

We derive here the exterior multipole wave solution (2.8) of the linearized form of equations (2.1).

In Galilean coordinates (x_α, t), the potential ϕ_i for outgoing waves in weak fields may be written in the Kirchhoff form

$$\phi_i = \int_V r^{*-1} J_i(\tilde{x}_\alpha, t - r^*)\,\mathrm{d}v \qquad \eta^{ab} J_{a,b} = 0 \tag{A.1}$$

(Eddington 1924, § 74, Rotenberg 1966), in which the integral covers any fixed space volume V containing the source of the field, and r^* is the distance of the point $\tilde{\mathrm{P}}(\tilde{x}_\alpha)$, associated with the space element $\mathrm{d}v = \mathrm{d}\tilde{x}_1\,\mathrm{d}\tilde{x}_2\,\mathrm{d}\tilde{x}_3$ of integration, from the field point $\mathrm{P}(x_\alpha)$ of interest. Carrying out the Taylor expansion about $(\tilde{x}_\alpha, t-r)$ for the integrand in the first of equations (A.1) we obtain

$$\frac{1}{r^*} J_i(\tilde{x}_\alpha, t - r^*) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{g^n}{r^*}\right) J_i^{(n)}(\tilde{x}_\alpha, t - r) \tag{A.2}$$

with

$$g \stackrel{\text{def}}{=} r^* - r \tag{A.3}$$

and the superscript symbol (n) denoting $\partial^n/\partial t^n$. Utilizing the binomial theorem in expanding g^n/r^* $(n = 0, 1, 2, \ldots)$ in ascending powers of $\tilde{r}/r$ corresponding to the range $r > \tilde{r} = \mathrm{O}\tilde{\mathrm{P}} = (\tilde{x}_\sigma \tilde{x}_\sigma)^{1/2}$, we find

$$\begin{gathered}\frac{1}{r^*} = \frac{1}{r}\sum_{n=0}^{\infty} \frac{\tilde{r}^n}{r^n} \mathrm{P}_n(\cos\theta^*) \qquad \frac{g}{r^*} = -\sum_{n=1}^{\infty} \frac{\tilde{r}^n}{r^n} \mathrm{P}_n(\cos\theta^*) \\ \frac{g^2}{r^*} = r\left\{\frac{\tilde{r}^2}{r^2}\cos^2\theta^* + \mathrm{O}\left[\left(\frac{\tilde{r}}{r}\right)^3\right]\right\} \ldots \frac{g^n}{r^*} = r^{n-1}\mathrm{O}\left[\left(\frac{\tilde{r}}{r}\right)^n\right] \ldots\end{gathered} \tag{A.4}$$

where θ^* is the angle $\mathrm{PO}\tilde{\mathrm{P}}$ and P_n are the Legendre polynomials. Substituting the expansions (A.4) into the expansion (A.2), employing the formulae

$$\tilde{r}^2 = \tilde{x}_\sigma \tilde{x}_\sigma \qquad \cos\theta^* = n_\sigma \tilde{x}_\sigma/\tilde{r} \qquad (n_\sigma = x_\sigma/r), \tag{A.5}$$

inserting the result in the first of equations (A.1) and adopting the notations (2.3), (2.5) and (2.6), we arrive at the following exterior multipole wave solution for ϕ_i:

$$\begin{aligned}\phi_\alpha &= e[ar^{-1}h_\alpha + a^2 n_\sigma(r^{-1}l'_{\alpha\sigma} + r^{-2}l_{\alpha\sigma}) + \mathrm{O}(a^3)] \\ \phi_4 &= e\{r^{-1} + an_\sigma(r^{-1}k'_\sigma + r^{-2}k_\sigma) \\ &\quad + a^2[\tfrac{1}{2}r^{-1}n_\sigma n_\rho k''_{\sigma\rho} + \tfrac{1}{2}(3n_\sigma n_\rho - \delta_{\sigma\rho})(r^{-2}k'_{\sigma\rho} + r^{-3}k_{\sigma\rho})] + \mathrm{O}(a^3)\};\end{aligned} \tag{A.6}$$

h_α, k_α, $k_{\alpha\beta}$ and $l_{\alpha\beta}$ are functions of $t-r$ and a prime indicates differentiation with respect to this argument.

To convert the solution (A.6) to the form (2.8), it is necessary to obtain certain relations among the leading $h_{i:\sigma\rho\tau\ldots}$ (see relations (A.9) and (A.11) below). This is done by applying a standard treatment on the electromagnetic conservation equation (2.4), here written as

$$J_{4,4} = J_{\alpha,\alpha} \tag{A.7}$$

similar to the one applied to the gravitational conservation equations $\eta^{ab}T_{ia,b} = 0$.

Multiplying equation (A.7) by x_β and integrating over any fixed space volume V containing the source we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V x_\beta J_4\,\mathrm{d}v = \int_V x_\beta J_{4,4}\,\mathrm{d}v = \int_V x_\beta J_{\alpha,\alpha}\,\mathrm{d}v = \int_V (x_\beta J_\alpha)_{,\alpha}\,\mathrm{d}v - \int_V J_\beta\,\mathrm{d}v. \tag{A.8}$$

By virtue of Gauss's theorem and the fact that $J_i = 0$ on the boundary S of V, the first integral on the extreme right of equation (A.8) vanishes, and so

$$\int_V J_\beta\,\mathrm{d}v = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V x_\beta J_4\,\mathrm{d}v$$

which in the notations (2.3), (2.5) and (2.6) gives

$$h_\alpha = -k'_\alpha. \tag{A.9}$$

Similarly, multiplying equation (A.7) by $x_\beta x_\gamma$ and integrating over V we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V x_\beta x_\gamma J_4\,\mathrm{d}v = \int_V x_\beta x_\gamma J_{4,4}\,\mathrm{d}v = \int_V x_\beta x_\gamma J_{\alpha,\alpha}\,\mathrm{d}v$$

$$= \int_V (x_\beta x_\gamma J_\alpha)_{,\alpha}\,\mathrm{d}v - \int_V (x_\gamma J_\beta + x_\beta J_\gamma)\,\mathrm{d}v. \tag{A.10}$$

On account of Gauss's theorem, the first integral on the extreme right of equation (A.10) vanishes, and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V x_\beta x_\gamma J_4\,\mathrm{d}v = -\int_V (x_\gamma J_\beta + x_\beta J_\gamma)\,\mathrm{d}v$$

which in the notations (2.3), (2.5) and (2.6) yields

$$k'_{\alpha\beta} = -l_{\alpha\beta} - l_{\beta\alpha}. \tag{A.11}$$

The relations (A.9) and (A.11) immediately transform the multipole wave solution (A.6) into the required form (2.8).

Appendix 2. Integral formulae for the energy and angular momentum fluxes

To establish the expressions (5.6) and (5.7) for the rates of outward flow of energy and angular momentum in the linear approximation, we start with the well known formulae

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \lim_{r\to\infty}\int_S E^{4\alpha} n_\alpha\,\mathrm{d}S \tag{A.12}$$

$$M_\alpha = -\epsilon_{\alpha\beta\gamma}\lim_{r\to\infty}\int_V x_\beta E^{\gamma 4}\,\mathrm{d}v \tag{A.13}$$

in Galilean coordinates x_i, where S is a large sphere, with centre O and radius r, enclosing volume V and $\epsilon_{\alpha\beta\gamma}$ is the permutation symbol.

Let us first consider the formula (A.13) to deduce equations (5.7). Differentiating it with respect to t we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(M_\alpha) = -\epsilon_{\alpha\beta\gamma}\lim_{r\to\infty}\int_V x_\beta E^{\gamma 4}{}_{,4}\,\mathrm{d}v = \epsilon_{\alpha\beta\gamma}\lim_{r\to\infty}\int_V x_\beta E^{\gamma\delta}{}_{,\delta}\,\mathrm{d}v \tag{A.14}$$

in the linear approximation, since in this approximation

$$E^{ik}{}_{,k} = 0 \equiv E^{\gamma\delta}{}_{,\delta} = -E^{\gamma 4}{}_{,4} \tag{A.15}$$

where no 4-current J_i is present. Hence, in the linear approximation, equation (A.14) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(M_\alpha) = \epsilon_{\alpha\beta\gamma} \lim_{r\to\infty} \int_V [(x_\beta E^{\gamma\delta})_{,\delta} - E^{\gamma\beta}]\,\mathrm{d}v = \epsilon_{\alpha\beta\gamma} \lim_{r\to\infty} \int_V (x_\beta E^{\gamma\delta})_{,\delta}\,\mathrm{d}v \tag{A.16}$$

since $\epsilon_{\alpha\beta\gamma} E^{\beta\gamma} = 0$ ($E^{\beta\gamma}$ being symmetric and $\epsilon_{\alpha\beta\gamma}$ being skew). By Gauss's theorem this yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(M_\alpha) = \epsilon_{\alpha\beta\gamma} \lim_{r\to\infty} \int_S x_\beta E^{\gamma\delta} n_\delta\,\mathrm{d}S = \epsilon_{\alpha\beta\gamma} \lim_{r\to\infty} r^3 \int n_\beta n_\delta E^{\gamma\delta}\,\mathrm{d}\Omega \tag{A.17}$$

from the definitions (2.7) and (5.8). On transformation from Galilean coordinates $x_i = (x, y, z, t)$ to the Sachs coordinates $\bar{x}_i = (r, \theta, \phi, u)$, equation (A.17) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(M_\alpha) &= \epsilon_{\alpha\beta\gamma} \lim_{r\to\infty} r^3 \int n_\beta n_\delta \bar{E}^{\sigma\rho} \frac{\partial x_\gamma}{\partial \bar{x}_\sigma} \frac{\partial x_\delta}{\partial \bar{x}_\rho}\,\mathrm{d}\Omega \\ &= \epsilon_{\alpha\beta\gamma} \lim_{r\to\infty} r^3 \int n_\beta \left(\frac{x_\delta}{r}\right) \bar{E}^{\sigma\rho} \frac{\partial x_\gamma}{\partial \bar{x}_\sigma} \frac{\partial x_\delta}{\partial \bar{x}_\rho}\,\mathrm{d}\Omega \\ &= \epsilon_{\alpha\beta\gamma} \lim_{r\to\infty} r^2 \int n_\beta \left(x_\delta \frac{\partial x_\delta}{\partial \bar{x}_\rho}\right) \bar{E}^{\sigma\rho} \frac{\partial x_\gamma}{\partial \bar{x}_\sigma}\,\mathrm{d}\Omega\end{aligned}$$

which, by virtue of the fact that

$$x_\delta \frac{\partial x_\delta}{\partial \bar{x}_a} = \frac{1}{2} \frac{\partial (x_\delta x_\delta)}{\partial \bar{x}_a} = \frac{1}{2} \frac{\partial (r^2)}{\partial \bar{x}_a} = r \frac{\partial r}{\partial \bar{x}_a} = r \delta_a^1, \tag{A.18}$$

gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(M_\alpha) = \epsilon_{\alpha\beta\gamma} \lim_{r\to\infty} r^3 \int \bar{E}^{\sigma 1} n_\beta \frac{\partial x_\gamma}{\partial \bar{x}_\sigma}\,\mathrm{d}\Omega. \tag{A.19}$$

Use of equations (2.7) and (4.6) in equation (A.19) leads eventually to equations (5.7).

To deduce equation (5.6), we use in equation (A.12) the transformation law from Galilean to the Sachs coordinates as above, the result (A.18) and equations (4.6).

The bars referring to the Sachs coordinates are no longer required in equations (5.6) and (5.7) as symbols of distinction.

We conclude by explaining the techniques with which the calculations of the expressions (5.9) and (5.10) from the formulae (5.6) and (5.7), via the values for E_{ik} specified by equations (5.1) to (5.5), may be significantly reduced. The most complicated computation results from calculating $\mathrm{d}\overset{(22)}{M^{\mathrm{G}}_\alpha}/\mathrm{d}t$; so we confine attention to the evaluation of the component $\mathrm{d}\overset{(22)}{M^{\mathrm{G}}_1}/\mathrm{d}t$ to illustrate the reduction in calculation.

From the first of equations (5.7) and equations (3.4), (4.5), (5.4) and (5.5) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overset{(22)}{M^{\mathrm{G}}_1}) = (l''_{\rho\sigma} - 2k'_\sigma k''_\rho) \int n_\sigma n_\rho\,\mathrm{d}\Omega \tag{A.20}$$

where

$$p_\alpha \overset{\text{def}}{=} n_{\alpha,2} \sin\phi + n_{\alpha,3} \cos\theta \cos\phi. \qquad \text{(A.21)}$$

It is readily seen from equation (2.7) that

$$p_\alpha = (0, n_3, -n_2). \qquad \text{(A.22)}$$

So

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overset{(22)}{M_1^{\mathrm{G}}}) = (l''_{2\sigma} - 2k'_\sigma k''_2)\int n_\sigma n_3 \,\mathrm{d}\Omega + (-l''_{3\sigma} + 2k'_\sigma k''_3)\int n_\sigma n_2 \,\mathrm{d}\Omega$$

which from the easily shown result that

$$\int n_\alpha n_\beta \,\mathrm{d}\Omega = \tfrac{4}{3}\pi\delta_{\alpha\beta} \qquad \text{(A.23)}$$

yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overset{(22)}{M_1^{\mathrm{G}}}) = \tfrac{4}{3}\pi\epsilon_{1\sigma\rho}(l''_{\sigma\rho} + 2k'_\sigma k''_\rho). \qquad \text{(A.24)}$$

Since we are interested solely in the secular change in $\overset{(22)}{M_1^{\mathrm{G}}}$ we may ignore the term $l''_{\sigma\rho}$ in the parentheses on the right of equation (A.24), since this term only has a periodic effect on $\overset{(22)}{M_1^{\mathrm{G}}}$, and write

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overset{(22)}{M_1^{\mathrm{G}}}) = \tfrac{8}{3}\pi\epsilon_{1\sigma\rho}k'_\sigma k''_\rho \qquad \text{(A.25)}$$

in agreement with equation (5.10).

Appendix 3. The approximate field equations for the Sachs metric and their solution

Inserting the expansions (4.4) and (3.5) in the first of equations (1.1) we obtain the (*ps*) approximation, in coordinates of the Sachs metric (4.1), as the ten equations below (in which $R'_{ik} \overset{\text{def}}{=} R_{ik} + 8\pi E_{ik}$). To save printing, the symbols (*ps*), which ought to have been placed above the capital letters, have been omitted throughout this appendix, except where confusion may result without them.

$$2R'_{11} = 0: \quad -4r^{-1}F_1 = P \qquad \text{(A.26)}$$

$$\begin{aligned} 2r^{-2}R'_{22} = 0: \quad & B_{11} - 2B_{14} + 2r^{-1}(B_1 - B_4 + D_1 - F_1 - G_{12}) \\ & + r^{-2}(-B_{22} + B_{33}\operatorname{cosec}^2\theta - 3B_2\cot\theta + 2B + 2D + 2F_{22} - 4F - 4G_2 \qquad \text{(A.27)} \\ & - 2G\cot\theta + 2I_{23}\operatorname{cosec}\theta + 2I_3\operatorname{cosec}\theta\cot\theta - 2J_3\operatorname{cosec}\theta) = Q \end{aligned}$$

$2r^{-2}\operatorname{cosec}^2\theta R'_{33} = 0$:

$$-B_{11}+2B_{14}+2r^{-1}(-B_1+B_4+D_1-F_1-G_1\cot\theta-J_{13}\operatorname{cosec}\theta)$$
$$+r^{-2}(-B_{22}+B_{33}\operatorname{cosec}^2\theta-3B_2\cot\theta+2B+2D$$
$$+2F_{33}\operatorname{cosec}^2\theta+2F_2\cot\theta-4F-2G_2-4G\cot\theta$$
$$+2I_{23}\operatorname{cosec}\theta+2I_3\operatorname{cosec}\theta\cot\theta-4J_3\operatorname{cosec}\theta) = R \quad \text{(A.28)}$$

$$2R'_{44} = 0: \quad -D_{11}+2F_{14}+2r^{-1}(-D_1-D_4+2F_4+G_{24}+G_4\cot\theta+J_{34}\operatorname{cosec}\theta)$$
$$-r^{-2}(D_{22}+D_{33}\operatorname{cosec}^2\theta+D_2\cot\theta) = S \quad \text{(A.29)}$$

$$2r^{-1}R'_{12} = 0: -G_{11}+r^{-1}(-B_{12}-2B_1\cot\theta+F_{12}-2G_1+I_{13}\operatorname{cosec}\theta)$$
$$+2r^{-2}(-F_2+G) = L \quad \text{(A.30)}$$

$$2R'_{14} = 0: \quad -D_{11}+2F_{14}+r^{-1}(-2D_1+G_{12}+G_1\cot\theta+J_{13}\operatorname{cosec}\theta)$$
$$+r^{-2}(-F_{22}-F_{33}\operatorname{cosec}^2\theta-F_2\cot\theta$$
$$+G_2+G\cot\theta+J_3\operatorname{cosec}\theta) = M \quad \text{(A.31)}$$

$$2r^{-1}R'_{24} = 0: -G_{11}+G_{14}$$
$$+r^{-1}(-B_{24}-2B_4\cot\theta-D_{12}+F_{12}+F_{24}-2G_1-G_4+I_{34}\operatorname{cosec}\theta)$$
$$+r^{-2}(-G_{33}\operatorname{cosec}^2\theta+J_{23}\operatorname{cosec}\theta+J_3\operatorname{cosec}\theta\cot\theta) = N \quad \text{(A.32)}$$

$2r^{-1}\operatorname{cosec}\theta\, R'_{13} = 0$:

$$-J_{11}+r^{-1}(B_{13}\operatorname{cosec}\theta+F_{13}\operatorname{cosec}\theta+I_{12}+2I_1\cot\theta-2J_1)$$
$$+2r^{-2}(-F_3\operatorname{cosec}\theta+J) = U \quad \text{(A.33)}$$

$2r^{-2}\operatorname{cosec}\theta\, R'_{23} = 0$:

$$-I_{11}+2I_{14}+r^{-1}(-G_{13}\operatorname{cosec}\theta-2I_1+2I_4-J_{12}+J_1\cot\theta)$$
$$+r^{-2}(2F_{23}\operatorname{cosec}\theta-2F_3\operatorname{cosec}\theta\cot\theta$$
$$-G_3\operatorname{cosec}\theta-J_2+J\cot\theta) = V \quad \text{(A.34)}$$

$2r^{-1}\operatorname{cosec}\theta R'_{34} = 0$:

$$\begin{aligned} &-J_{11}+J_{14}+r^{-1}(B_{34}\operatorname{cosec}\theta-D_{13}\operatorname{cosec}\theta+F_{13}\operatorname{cosec}\theta \\ &+F_{34}\operatorname{cosec}\theta+I_{24}+2I_4\cot\theta-2J_1-J_4) \\ &+r^{-2}(G_{23}\operatorname{cosec}\theta-G_3\operatorname{cosec}\theta\cot\theta \\ &-J_{22}-J_2\cot\theta+J\operatorname{cosec}^2\theta) = W. \end{aligned} \tag{A.35}$$

Here, a subscript 1, 2, 3 or 4 after B, D, F, G, I or J denotes differentiation with respect to r, θ, ϕ or u respectively—a notation to apply to any non-tensorial symbol, unless implied otherwise in the context. The second of equations (4.1) has been made use of; thus C does not appear in the above equations. The terms linear in $\overset{(ps)}{g}_{ik}$ and their derivatives appear explicitly on the left of these equations, while the terms nonlinear in $\overset{(qr)}{g}_{ik}$ and their derivatives, known from earlier approximations, accompany $\overset{(ps)}{E}_{ik}$ to form the quantities $P, \ldots, W$ on the right.

It can be shown that the following six equations can be derived from the first seven equations of the above (ps) approximation; a proof is to appear in a later work:

$$F = -\frac{1}{4}\int rP\,\mathrm{d}r+\eta(\theta, \phi, u) \tag{A.36}$$

$$\begin{aligned} \Box' D &\overset{\text{def}}{=} D_{11}-2D_{14}+2r^{-1}(D_1+D_4)+r^{-2}(D_{22}+D_2\cot\theta+D_{33}\operatorname{cosec}^2\theta) \\ &= -S+2(F_1+2r^{-1}F+r^{-2}X)_4 \end{aligned} \tag{A.37}$$

$$\begin{aligned} \Box'' G &\overset{\text{def}}{=} r(G_{111}-2G_{114})+(3G_{11}-2G_{14}) \\ &\quad +r^{-1}[G_{122}+3G_{12}\cot\theta+G_1(\cot^2\theta-1)+G_{133}\operatorname{cosec}^2\theta+2G_4] \\ &\quad +r^{-2}[-G_{22}-3G_2\cot\theta+G(1-\cot^2\theta)-G_{33}\operatorname{cosec}^2\theta] \\ &= rL_4-(rN)_1+2D_{11}\cot\theta+F_{112}+2r^{-1}F_{24}+[r^{-2}(X_2+2X\cot\theta)]_1 \end{aligned} \tag{A.38}$$

$$J = \sin\theta\int(rD_1-G_2+r^{-1}X)\,\mathrm{d}\phi-\cos\theta\int G\,\mathrm{d}\phi+\tau(r, \theta, u) \tag{A.39}$$

$$\begin{aligned} \Box''' B &\overset{\text{def}}{=} B_{11}-2B_{14}+2r^{-1}(B_1-B_4) \\ &= \tfrac{1}{2}(Q-R)-M-D_{11}+2F_{14}+2r^{-1}(-D_1+G_{12})+2r^{-2}(-F_{22}+G_2) \end{aligned} \tag{A.40}$$

$$\begin{aligned} I = \sin\theta &\int\left(\int[rL+2r^{-1}(F_2-G)]\,\mathrm{d}r+(rG)_1+B_2-F_2\right)\mathrm{d}\phi \\ &+2\cos\theta\int B\,\mathrm{d}\phi+\mu(\theta, \phi, u)+\nu(r, \theta, u) \end{aligned} \tag{A.41}$$

where

$$X \overset{\text{def}}{=} \int[r^2(M-2F_{14})+(F_{22}+F_2\cot\theta+F_{33}\operatorname{cosec}^2\theta)]\,\mathrm{d}r+\chi(\theta, \phi, u) \tag{A.42}$$

and

$$\eta(\theta, \phi, u) \qquad \chi(\theta, \phi, u) \qquad \tau(r, \theta, u) \qquad \mu(\theta, \phi, u) \qquad \nu(r, \theta, u) \tag{A.43}$$

are five functions of integration. This set of equations will be considered as the formal solution of the (*ps*) approximation. For F is known at once from equation (A.36), so that the right-hand side of equation (A.37) is readily computed. The differential equation (A.37) can be solved for D by using, as a trial solution, an expansion in ascending powers of r^{-1}. Subsequently the differential equation (A.38) can be solved for G by a similar process, as the terms on the right of this equation are now known. The results so far allow J to be calculated immediately from equation (A.39), and the differential equation (A.40) to be solved for B by use of a trial solution similar to that for D or G. This in turn enables I to be evaluated at once from equation (A.41).

During the foregoing process of solution, values for the five arbitrary functions (A.43) of integration and complementary solutions of the differential equations (A.37), (A.38) and (A.40) must be chosen with extreme caution so that the (*ps*) solution involving them satisfies all the ten (*ps*) field equations (A.26) to (A.35), including the last three, and is regular for all θ, ϕ and u and for all $r > 0$. It is desirable that the (*ps*) solution also satisfies the Galilean conditions at spatial infinity. However, no additional complementary functions are to be used towards obtaining the (*ps*) solution, since these constitute gravitational source functions of the linear approximation to $R_{ik} = 0$, which, from what was stated in § 3, are not our concern here.

Appendix 4. The Schwarzschild angular momentum field

We outline here the derivation of the metric (6.11), to the accuracy indicated, for a rigid spherical body of mass m rotating with angular momentum $A_{\alpha\beta}$.

In Rotenberg (1972a) it is shown that the linearized form of the metric of Sachs type representing the external field of any isolated coherent gravitational source has a quadrupole wave—immediately following the Schwarzschild part

$$-r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)+(1-2mr^{-1})\,\mathrm{d}u^2+2\,\mathrm{d}r\,\mathrm{d}u \tag{A.44}$$

—of the form

$$\begin{aligned}(n_{\sigma,2}n_{\rho,2}-s^{-2}n_{\sigma,3}n_{\rho,3})&(r^{-1}M''_{\sigma\rho}+r^{-3}M_{\sigma\rho})(-r^2\,\mathrm{d}\theta^2+r^2\sin^2\theta\,\mathrm{d}\phi^2)\\ &+(-3n_\sigma n_\rho+\delta_{\sigma\rho})(2r^{-1}M''_{\sigma\rho}+2r^{-2}M'_{\sigma\rho}+r^{-3}M_{\sigma\rho})\,\mathrm{d}u^2\\ &+n_\sigma n_{\rho,2}[2r^{-1}M''_{\sigma\rho}+2r^{-2}(-2M'_{\sigma\rho}+A_{\sigma\rho})-3r^{-3}M_{\sigma\rho}](2r\,\mathrm{d}\theta\,\mathrm{d}u)\\ &-2s^{-1}n_{\sigma,2}n_{\rho,3}(r^{-1}M''_{\sigma\rho}+r^{-3}M_{\sigma\rho})(2r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi)\\ &+s^{-1}n_\sigma n_{\rho,3}[2r^{-1}M''_{\sigma\rho}+2r^{-2}(-2M'_{\sigma\rho}+A_{\sigma\rho})-3r^{-3}M_{\sigma\rho}]\\ &\times(2r\sin\theta\,\mathrm{d}\phi\,\mathrm{d}u).\end{aligned} \tag{A.45}$$

In this, $M_{\alpha\beta}$ are the second mass moments, at retarded time u, of the source about the Cartesian coordinate planes $x_\alpha = 0$. For a rigid sphere the time derivatives of $M_{\alpha\beta}$ vanish and the combination of the Schwarzschild and quadrupole metrics (A.44) and (A.45) reduces at once to the metric (6.11), in which terms of order r^{-3} have been ignored.

Appendix 5. Some useful identities

The following identities, involving n_α, and perhaps a few others deducible from these, are found useful in §§ 5 and 6, especially in the derivation and verification of the solutions (6.4) to (6.6). Not all the identities are independent of each other; some may be derived from others, but all of these can be directly checked via equation (2.7).

$$n_{\alpha,22}+n_{\alpha,2}\cot\theta+n_{\alpha,33}\operatorname{cosec}^2\theta = -2n_\alpha \tag{A.46}$$

$$n_{\alpha,22} = -n_\alpha \qquad n_{\alpha,23} = n_{\alpha,3}\cot\theta \equiv (n_{\alpha,2}-n_\alpha\cot\theta)_{,3} = 0 \tag{A.47}$$

$$n_{\alpha,2}\cot\theta+n_\alpha+n_{\alpha,33}\operatorname{cosec}^2\theta = 0 \tag{A.48}$$

$$(n_\alpha n_{\beta,2})_{,2}+n_\alpha n_{\beta,2}\cot\theta+(n_\alpha n_{\beta,3})_{,3}\operatorname{cosec}^2\theta = \delta_{\alpha\beta}-3n_\alpha n_\beta \tag{A.49}$$

$$(n_\alpha n_\beta)_{,22}+(n_\alpha n_\beta)_{,2}\cot\theta+(n_\alpha n_\beta)_{,33}\operatorname{cosec}^2\theta = 2(\delta_{\alpha\beta}-3n_\alpha n_\beta) \tag{A.50}$$

$$\begin{aligned} n_{\alpha,2}n_\beta-n_\alpha n_{\beta,2} &= -(n_{\alpha,2}n_{\beta,3}-n_{\alpha,3}n_{\beta,2})_{,3}\operatorname{cosec}^2\theta \\ &= -(n_{\alpha,3}n_\beta-n_\alpha n_{\beta,3})_{,3}\operatorname{cosec}\theta\sec\theta \end{aligned} \tag{A.51}$$

$$(n_\alpha n_\beta)_{,2}+2(n_{\alpha,2}n_{\beta,2}-n_{\alpha,3}n_{\beta,3}\operatorname{cosec}^2\theta)\cot\theta+(n_{\alpha,2}n_{\beta,3}+n_{\alpha,3}n_{\beta,2})_{,3}\operatorname{cosec}^2\theta = 0 \tag{A.52}$$

$$(\delta_{\alpha\beta}-2n_\alpha n_\beta-n_{\alpha,2}n_{\beta,2})-n_\alpha n_{\beta,2}\cot\theta-(n_\alpha n_{\beta,3})_{,3}\operatorname{cosec}^2\theta = 0 \tag{A.53}$$

$$n_{\alpha,2}n_{\beta,2}+(n_{\alpha,3}n_{\beta,3})\operatorname{cosec}^2\theta = \delta_{\alpha\beta}-n_\alpha n_\beta \tag{A.54}$$

$$S_{\alpha\beta} \stackrel{\text{def}}{=} \delta_{\alpha\beta}-n_\alpha n_\beta-2n_{\alpha,2}n_{\beta,2} = -n_{\alpha,2}n_{\beta,2}+n_{\alpha,3}n_{\beta,3}\operatorname{cosec}^2\theta \tag{A.55}$$

$$Z_{\alpha\beta} \stackrel{\text{def}}{=} n_{\alpha,2}n_{\beta,3}+n_{\alpha,3}n_{\beta,2} = (n_\alpha n_\beta)_{,23}-(n_\alpha n_\beta)_{,3}\cot\theta \tag{A.56}$$

$$S_{\alpha\beta,2} = (n_\alpha n_\beta)_{,2} \tag{A.57}$$

$$S_{\alpha\beta,2}+2S_{\alpha\beta}\cot\theta-Z_{\alpha\beta,3}\operatorname{cosec}^2\theta = 2(n_\alpha n_\beta)_{,2} \tag{A.58}$$

$$Z_{\alpha\beta,2}+Z_{\alpha\beta}\cot\theta+S_{\alpha\beta,3} = -2(n_\alpha n_\beta)_{,3} \tag{A.59}$$

$$S_{\alpha\beta,22}+3S_{\alpha\beta,2}\cot\theta-2S_{\alpha\beta}-S_{\alpha\beta,33}\operatorname{cosec}^2\theta-2Z_{\alpha\beta,23}\operatorname{cosec}^2\theta = 4(\delta_{\alpha\beta}-3n_\alpha n_\beta). \tag{A.60}$$

If

$$\Box^*(\gamma(\theta,\phi)) \stackrel{\text{def}}{=} \gamma_{22}+3\gamma_2\cot\theta+(\gamma_{33}+\gamma)\operatorname{cosec}^2\theta \tag{A.61}$$

then

$$\Box^*(n_\alpha n_{\beta,2}) = 2[-(n_\alpha n_\beta)_{,2}+(\delta_{\alpha\beta}-3n_\alpha n_\beta)\cot\theta]. \tag{A.62}$$

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